On the equation $F(x) = f(f(x))$

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1 The Problem

Following Ed Pegg Jr (see the section “Material added 21 November 04” on *http://www.Mathpuzzle.com*) we want to discuss the problem:

$$\begin{cases} \text{which functions } F \text{ have a representation of the form } F(x) = f(f(x))? \\ \text{for a given } F\text{, how many are the solutions, and how to build them?} \end{cases}$$

We will confine ourselves to work in a polynomial framework, assuming that both F and f are polynomials. In fact the non-polynomial framework seems out of reach because unexpected non-polynomial solutions come out very often; see later on. On the other hand, we will allow complex coefficients for our polynomials F, f: as often happens, the real results easely follow from the complex ones; and the complex approach gives rise to more elegant results.
Remark *After reading a preliminary redaction of this paper, Ed sent me the feedback:*

One interesting trick to add: $(f(f(x)) - x)/(f(x) - x)$ always divides evenly

thus I added a last section, “Factorizations”, absent in the previous redaction.

2 Preliminary Remarks

Let us firstly deal with a somewhat anomalous case: the function $F(x) = x + c$ has the solution $f(x) = x + c/2$. If $c \neq 0$ we will see that this is the unique solution; however, if $c = 0$, for any a we can also choose $f(x) = a - x$.
We will also see that, in the polynomial framework, the case $F(x) = x$ is the unique one giving raise to infinitely many solutions. Remark that this F is singular even in the non-polynomial framework: for $x \neq 0$ also the functions $f(x) = a/x$ solve. A last remark on the non-polynomial case: the rational function $F(x) = 2x/(1 - x^2)$ has, for $|x| < 1$, the trascendental solution $f(x) = \tan\left(\sqrt{2} \cdot \arctan(x)\right)$.
Let us fix some notations. For f polynomial of degree k, the polynomial $f(f(x))$ has degree k^2; thus, for a suitable k, we can represent the given F and the

unknown f in the form:

$$F(x)=\sum_{j=0}^{k^2} A_j x^{k-j}; \quad f(x)=\sum_{j=0}^{k} a_j x^{k-j}; \qquad (A_0, a_0 \neq 0);$$

remark that we used capital letters A_j for the data, and small letters a_j for the unknown coefficients. In other terms: we would freely choose the k^2+1 coefficients of F, and impose the equation $F(x)=f(f(x))$ by using the $k+1$ coefficients of f. It is quite natural to expect that, if $k>1$, this is impossible; and in fact we will see that, generally speaking, once we fix the first $k+1$ coefficients $A_0,\ldots,A_k$ we have no more degree of freedom.

3 Results

Let us evaluate $f(f(x))$ modulo the lower order terms, say the terms of order strictly less than k^2-k; we will use the symbol "$\cong$" to denote that some lower order terms have been suppressed. Because of

$$\begin{cases} f(f(x)) \cong a_0 \cdot [f(x)]^k + a_1 \cdot [f(x)]^{k-1}\,; \\ a_0 \cdot [f(x)]^k \cong a_0 \cdot \left[\left(a_0 \cdot x^k\right)^k + k \cdot \left[a_0 \cdot x^k\right]^{k-1} \cdot \left[a_1 \cdot x^{k-1} + \cdots + a_k\right] \right]\,; \\ a_1 \cdot [f(x)]^{k-1} \cong a_1 \left[a_0 \cdot x^k\right]^{k-1} \end{cases}$$

we get:

$$f(f(x)) \cong a_0^{k+1} \cdot x^{k*k} + k \cdot \left[a_0 \cdot x^k\right]^{k-1} \cdot \left[a_1 \cdot x^{k-1} + \cdots + a_k\right] + a_1 \left[a_0 \cdot x^k\right]^{k-1}$$

The comparison between the leading coefficients of $F(x)$ and $f(f(x))$ gives:

$$A_0 = a_0^{k+1}; \qquad A_j = k \cdot a_0^k \cdot a_j \text{ for } j=1,\ldots,k-1; \qquad A_k = k \cdot a_0^k \cdot a_k + a_1 \cdot a_0^{k-1}.$$

In particular, from $a_0^{k+1}=A_0$, we have $k+1$ possible choices for a_0 (recall that $A_0 \neq 0$). Once a value for a_0 has been fixed:

- if $k=0$ we finished;
- if $k=1$, we need $a_1 = A_1/\left(a_0+1\right)$, to be discussed later on;
- if $k>1$ we must choose:

  $$a_j = A_j/\left(k \cdot a_0^k\right) \text{ for } j=1,\ldots,k-1;$$
  $$a_k = \left(A_1 - a_1 \cdot a_0^k\right)/\left(k \cdot a_0^k\right) = \left(A_k - A_1/k\right)/\left(k \cdot a_0^k\right)$$

  the last formula following from the already known form of a_1.

The case $k=1$, as already remarked, deserves a surprise: for $A_0 \neq 1$ we simply need to choose $a_1 = A_1/\left(a_0+1\right)$; the same formula remains true if $A_0=1$ and we choose $a_0=1$; however, for $A_0=1$, if we want to choose $a_0=-1$, we need $A_1=0$; then any value for a_1 is allowed.

Let us summarize the results in the complex framework. Searching for representable polynomials $F(x)$ of degree k, we have:

- If $k < 2$, for any choice of $F(x)$ other than $x + c$, there exist exactly $k + 1$ solutions.
- If $k \geq 2$ we can freely choose the first $k+1$ coefficients of F; this generates $k + 1$ functions f that automatically select theyr own "lower order part" for F.

Let us show how the last claim works by discussing the original example $F(x) = x^4 - 4x^3 + 8x + 2$ proposed by Ed Pegg. Our formulas force $f(x) = \omega \cdot x^2 - 2\omega \cdot x + 1 - 2\omega$, where ω denotes a third root of the unit. Accordingly we get $f(f(x)) = x^4 - 4x^3 + 8x + 5 - 3\omega$; thus we need $\omega = 1$, say $f(x) = x^2 - 2x - 1$.

A last remark concerns the real framework: let $F(x)$ be a real polynomial of degree k, other than $x + c$. Among the $k + 1$ complex-coefficients polynomials f we have built, the number of real f is just one if k is odd; and, for even k, is two or none, according to $A_0 > 0$ or $A_0 < 0$.

4 Factorizations

A somewhat different approach can be based on factorizations results; e.g. let us remark that:

$F'(x)/f'(x)$ *always divides evenly*

as obvious because of $F'(x) = f(f(x))' = f'(f(x)) \cdot f'(x)$.

Remark*Let x_0 be a (real or complex) value such that $f(x_0) = x_0$. Then:*

- $F'(x_0) = f'(x_0)^2$.
- $F''(x_0) = f''(x_0)f'(x_0)^2 + f'(x_0)f''(x_0)$
- *For $k > 2$ any term appearing in $F^{(k)}(x_0)$ contains a factor like $f^{(j)}(x_0)$ for a $j \geq 2$.*

Now let us set $G(x) := f(f(x)) - x$ and $g(x) := f(x) - x$. For any x_0 with $g(x_0) = 0$ one has $G(x_0) = 0$; if one has also $g'(x_0) = 0$, say $f'(x_0) = 1$ then (see the previous remark) $G'(x_0) = F'(x_0) - 1 = 0$; always due to the previous remark, if $g''(x_0) = 0$, say $f''(x_0) = 0$, then $G''(x_0) = 0$; and so on.

In other words, any (real or complex) root of g is also a root of G with at least the same molteplicity; this implies that $g(x)$ is a factor of $G(x)$; say:

$(f(f(x)) - x)/(f(x) - x)$ *always divides evenly.*

Let us see how these properties can be used, by treating again the example $F(x) = x^4 - 4x^3 + 8x + 2$: we confine ourselves to discuss the real case, thus the leading coefficients of f and f' must be 1 and 2, instead of ω and 2ω.

Because of $F'(x) = 4(x-1)(x-1-\sqrt{3})(x-1+\sqrt{3})$, for $f'(x)$ we have to choose one among $2(x - 1)$, $2(x - 1 - \sqrt{3})$ and $2(x - 1 + \sqrt{3})$.

Because of $F(x) - x = (x - 2)(x + 1)(x - \frac{3+\sqrt{13}}{2})(x - \frac{3-\sqrt{13}}{2})$ we have a priori six possible choices for the second degree polynomial $f(x) - x$; but only the choice $f(x) - x = (x - \frac{3+\sqrt{13}}{2})(x - \frac{3-\sqrt{13}}{2}) = x^2 - 3x - 1$ is compatible with the formulas we got for $f'(x)$; thus we end up with the already found solution $f(x) = x^2 - 2x - 1$.