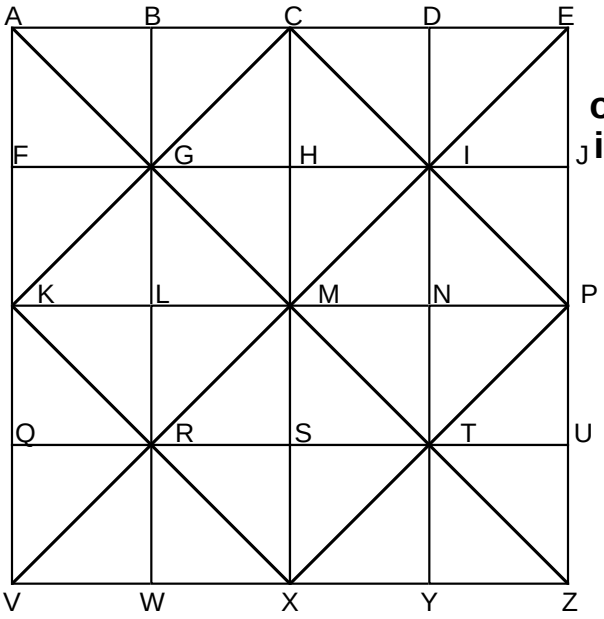
Problem from Weekend Edition - Sunday, December 5, 2010
Source: This was said to be the "King Ptolemy Problem" of Sam Loyd

How many triangles are there in the figure below?



Solution: (corrected twice)

There are 96 triangles (of positive area, that is) in the subdivided square.

FIRST SOLUTION: If one takes the lengths of the sides of the large square to be 4, then the area of the largest triangles, AEZ for example, is 8, while the area of the smallest, ABG for example, is 1/2.

With one exception, between these extremes, the areas double as one progresses from the smallest to the largest. The areas are 1/2, 1, 2, 4, 4.5 and 8.

Counting how many there are of each area is a convenient way to do the counting. There are 4 triangles of area 8, namely AEV, AEZ, AVZ, and EVZ. There are also 8 triangles of area 4, namely AEM, AMV, CKP, CKX, CPX, EMZ, KPX, and MVZ. Each of the 2 by 2 squares, such as ACMK has 4 triangles of area 2 and 4 of area 1. Thus, there are 16 each of such triangles of areas 2 and 1. In addition, there are 4 triangles of area 2 and 8 triangles of area 1 that span the second and third rows or columns (or both).

There are 4 3 by 3 squares that can be embedded into this 4 by 4 square and each of them has one diagonal, and on each side of it is a triangle or area 9/2. This produces 8 triangles of area 9/2. Finally, each of the 16 1 by 1 squares contains 2 triangles of area 1/2, so there are 32 such.

This gives a total of 4 + 8 + 8 + 16 + 4 + 8 + 16 + 32 = 96 triangles of areas 8, 4.5, 4, 2, 2, 1, 1 and 0.5 respectively.

SECOND SOLUTION: This time we will count how many triangles there are that can be contained in a 4x4, a 3x3, a 2x2, or a 1x1 subsquare.

There are 12 triangle that are contained in the entire 4x4 square but in no 3x3 square. These are the ones of area 8 or 4 listed above. There are 8 triangles that need at most a 3x3 square. These are the ones of area 9/2.

The ones that fit in a 1x1 square are the 32 of area 1/2. Those that fit in a 2x2 squre but not a 1x1 square are the most diverse. There are four nonoverlapping 2x2 squares in each corner. We willl analyze square ACMK and then quadruple the number of triangles in it to get the numbers in the others, This square contains 4 triangles of area 2 and 4 of area 1. In addition to these tringles that are contained in the outer 2x2 squares, there are those contained in the 2x2 square that are in rows and columns 2 and 3. There are 8 + 4 of these.

Together there are 4x8 + 4 + 8 = 44 triangles within 2x2 squares. All together, there are 12 + 8 + 44 +32 = 96 triangles.

THIRD SOLUTION: One can verify this number by listing all of the triangles in alphabetical order in terms of the labels on vertices. Such a list is given at the right. We again get 96

Triangles contained in this figure in alphabetic order on vertex labels. (The numbers in parentheses are their areas.)

1. ABG (1/2)
2. ACG (1)
3. ACK (2)
4. ACM (2)
5. ADT (9/2)
6. AEM (4)
7. AEV (8)
8. AEZ (8)
9. AFG (1/2)
10. AGK (1)
11. AKM (2)
12. AMV (4)
13. AQT (9/2)
14. AVZ (8)
15. BCG (1/2)
16. BER (9/2)
17. CDI (1/2)
18. CEI (1)
19. CEM (2)
20. CEP (2)
21. CGH (1/2)
22. CGI (1)
23. CGM (1)
24. CHI (1/2)
25. CIM (1)
26. CKM (2)
27. CKP (4)
28. CKX (4)
29. CMP (2)
30. CPX (4)
31. DEI (1/2)
32. EIJ (1/2)
33. EIP (1)
34. EMP (2)
35. EMZ (4)
36. ERU (9/2)
37. EVZ (8)
38. FGK (1/2)
39. FIV (9/2)
40. GHM (1/2)
41. GIM (2)
42. GIR (2)
43. GIT (2)
44. GJZ (9/2)
45. GKL (1/2)
46. GKM (1)
47. GKR (1)
48. GLM (1/2)
49. GMR (1)
50. GRT (2)
51. GWZ (9/2)
52. HIM (1/2)
53. IJP (1/2)
54. IMN (1/2)
55. IMP (1)
56. IMT (1)
57. INP (1/2)
58. IPT (1)
59. IRT (2)
60. IVY (9/2)
61. KLR (1/2)
62. KMR (1)
63. KMV (2)
64. KMX (2)
65. KPX (4)
66. KQR (1/2)
67. KRV (1)
68. KVX (2)
69. LMR (1/2)
70. MNT (1/2)
71. MPT (1)
72. MPX (2)
73. MPZ (2)
74. MRS (1/2)
75. MRT (1)
76. MRX (1)
77. MST (1/2)
78. MTX (1)
79. MVX (2)
80. MVZ (4)
81. MXZ (2)
82. NPT (1/2)
83. PTU (1/2)
84. P TZ (1)
85. PXZ (2)
86. QRV (1/2)
87. RSX (1/2)
88. RTX (1)
89. RVW (1/2)
90. RVX (1)
91. RWX (1/2)
92. STX (1/2)
93. TUZ (1/2)
94. TXY (1/2)
95. TXZ (1)
96. TYZ (1/2)