Fusible Numbers

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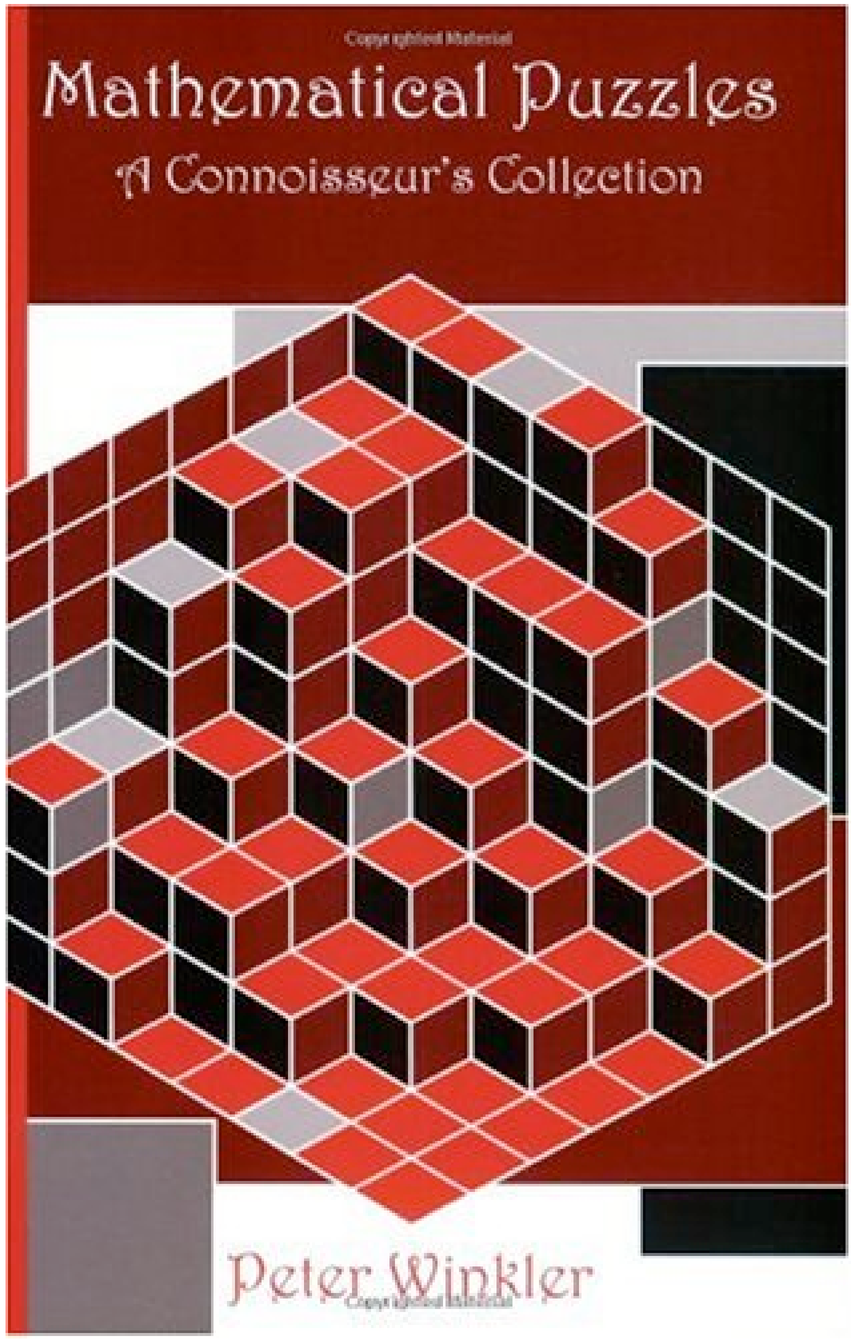
An old(?) puzzle

- You are given several fuses, each of which burn for exactly one minute.
- The fuses burn don't burn uniformly, so you can't predict how much fuse will be left after (say) 15 seconds.
- How do you measure an interval of 45 seconds?

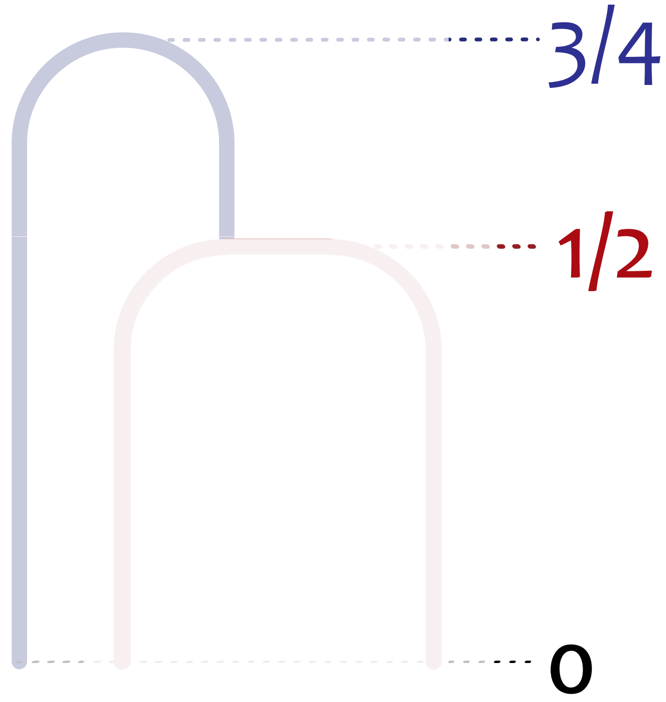
[Attributed to Carl Morris]



Solution

Measuring 30 seconds:

- Light both ends of a fuse simultaneously!
- The fuse burns out 30 seconds later.



Measuring 45 seconds:

- We need two fuses.
- Simultaneously light both ends of fuse A and one end of fuse B.
- Fuse A burns out 30 seconds later; light the other end of fuse B.
- Fuse B burns out 15 seconds later.

What else can we do with this?

Fusible numbers

Say that a real number *x is **fusible*** if one can measure *x* minutes exactly, using a finite number of 1-minute fuses.

- Fuses can be lit either at time 0, or precisely when another fuse burns out.
- Any finite number of fuse ends may be lit simultaneously at these times.
- The interval starts when first fuse is lit, ends when last fuse goes out.
- **No cheating!** Fuses can't be cut, stopped, or lit in the middle; no other clocks.

More formally...

If we light one end at time a and the other at time b, where $|a-b|<1$, the fuse burns out at time $\boldsymbol{a \sim b} := (a+b+1)/2$. (pronounced "$a$ fuse b")

A number x is ***fusible*** if and only if

- $x = 0$ or
- $x = a \sim b$ for some fusible numbers a and b with $|a-b|<1$

Small examples

$a \sim b := (a+b+1)/2$

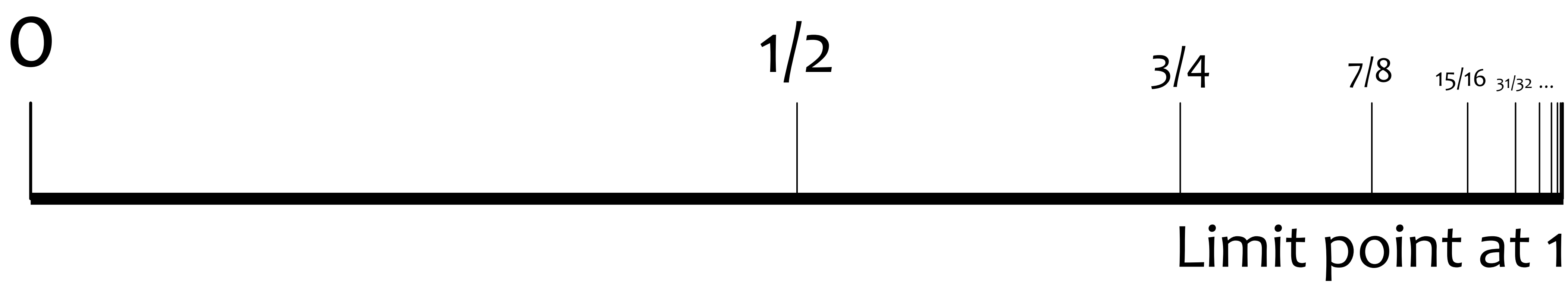
$$0 \sim 0 = 1/2$$

$$0 \sim 1/2 = 3/4$$

$$0 \sim 3/4 = 7/8$$

$$\vdots$$

$$0 \sim (1-2^{-n}) = 1-2^{-(n+1)}$$



Small examples

$a \sim b := (a+b+1)/2$

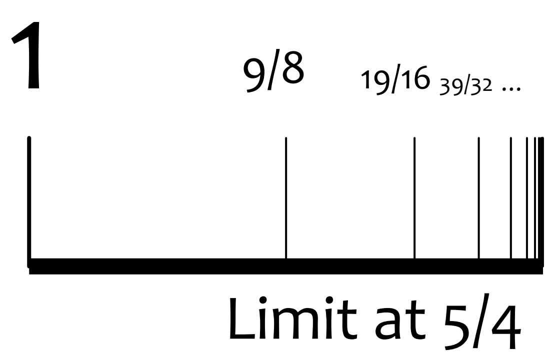
$$1/2 \sim 1/2 = 1$$

$$1/2 \sim 3/4 = 9/8$$

$$1/2 \sim 7/8 = 19/16$$

$$\dots$$

$$1/2 \sim (1-2^{-n}) = 5/4-2^{-(n+1)}$$



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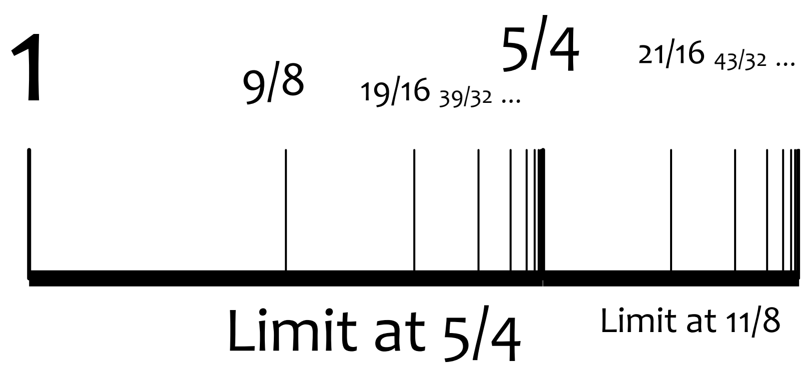

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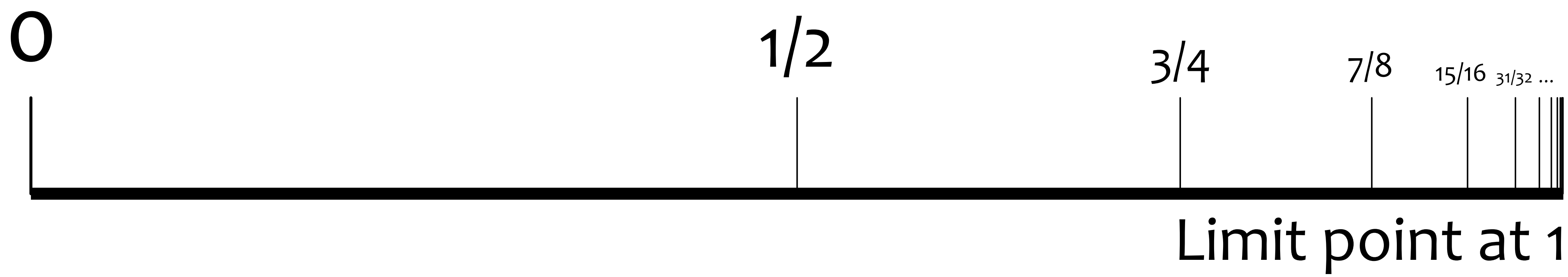
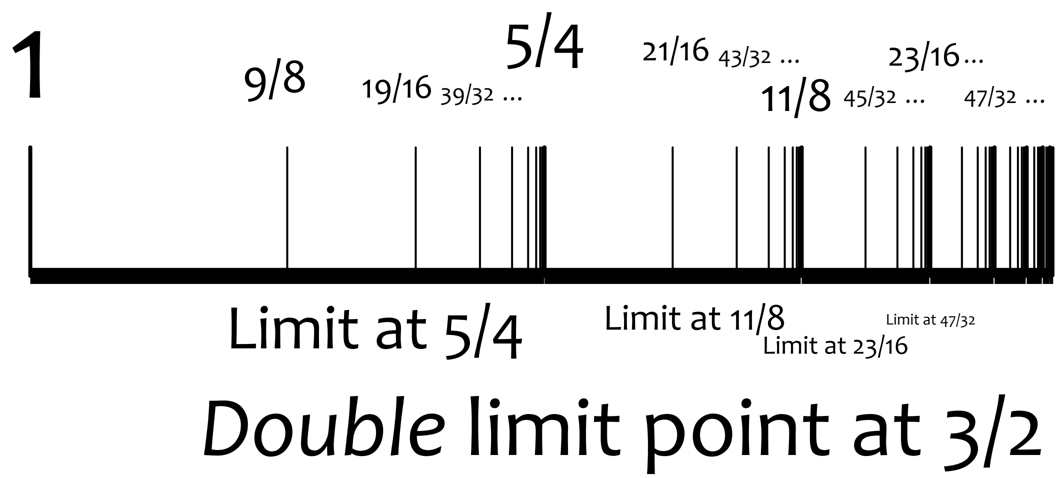
$$1/2 \sim (5/4-2^{-n}) = 11/8-2^{-(n+1)}$$

$$1/2 \sim (3/2-2^{-m}-2^{-n}) = 3/2-2^{-m}-2^{-(n+1)}$$

Double limit point at 3/2

Small examples





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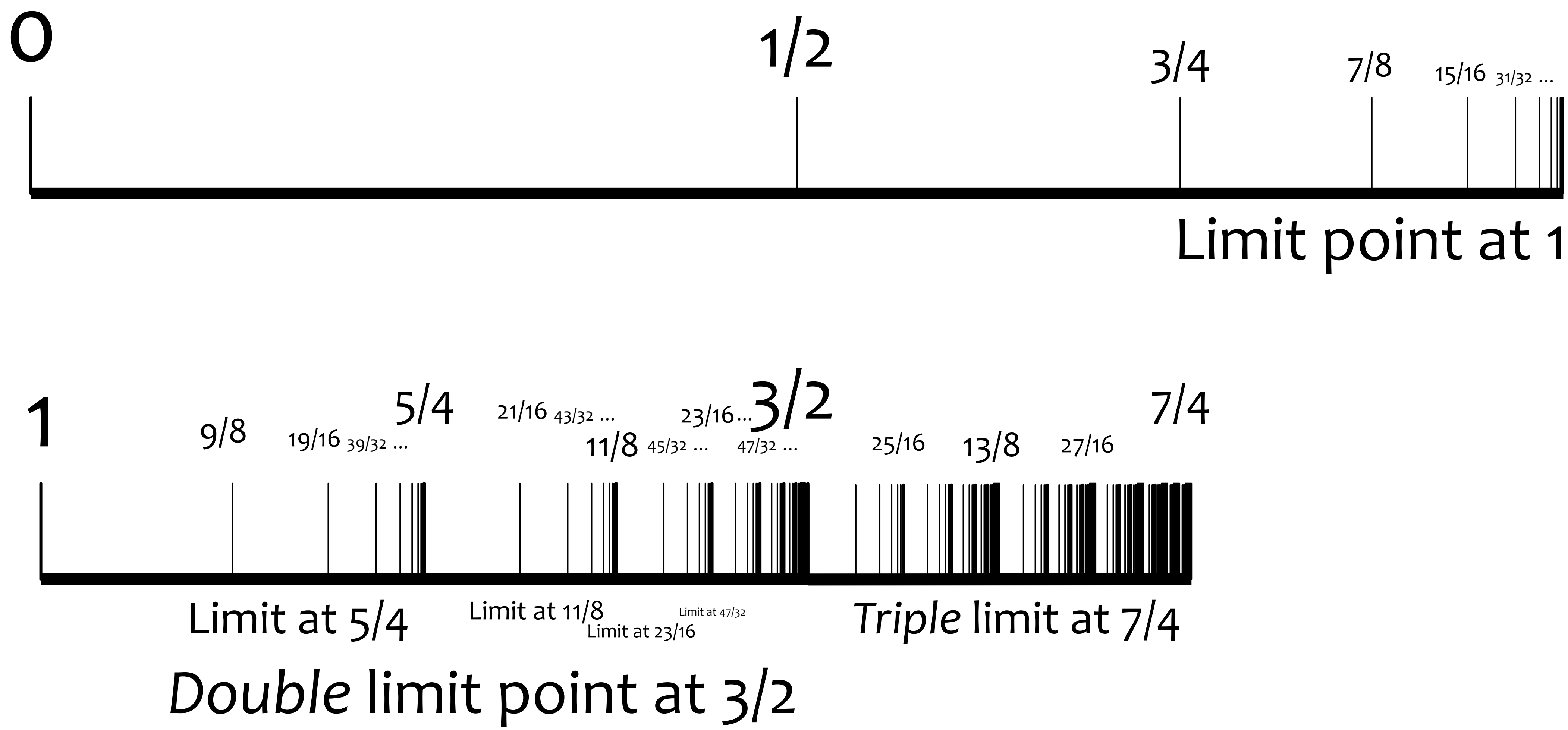

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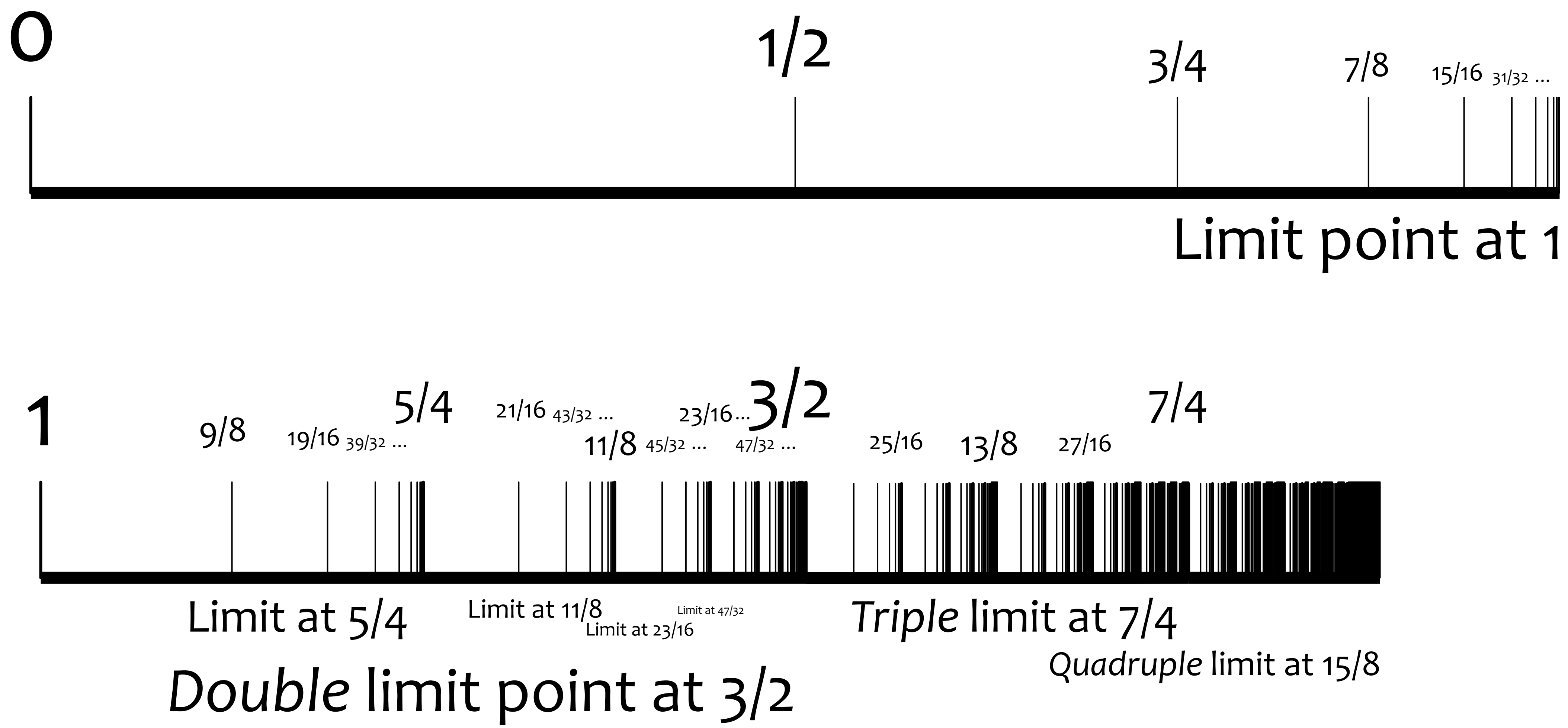

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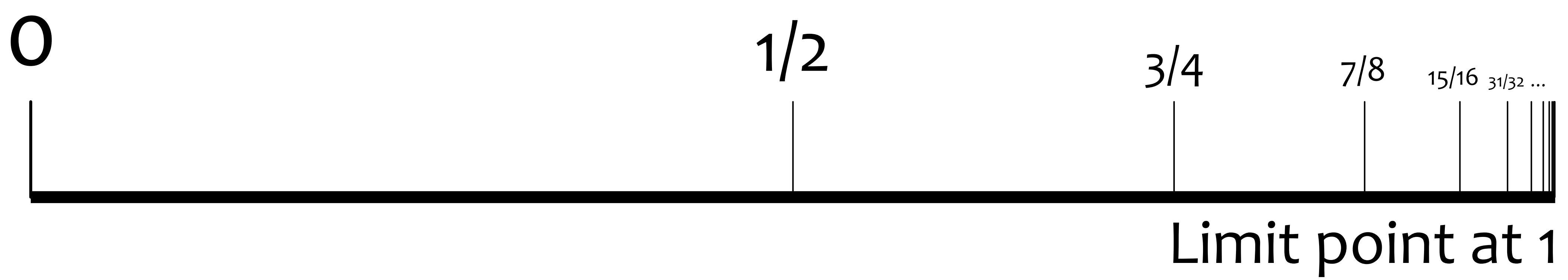
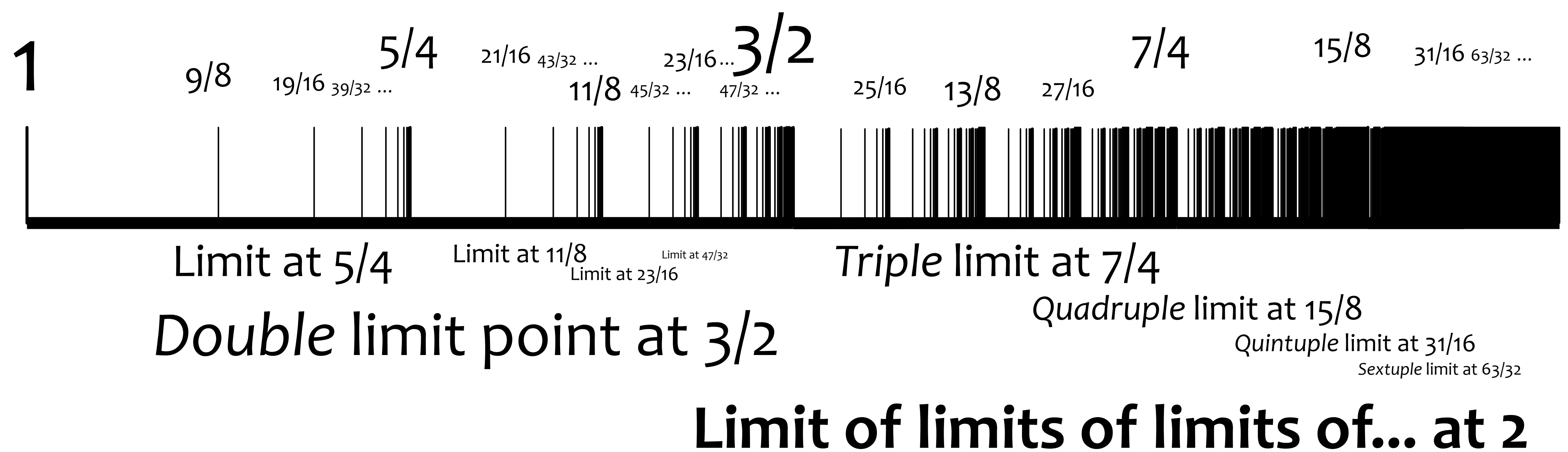
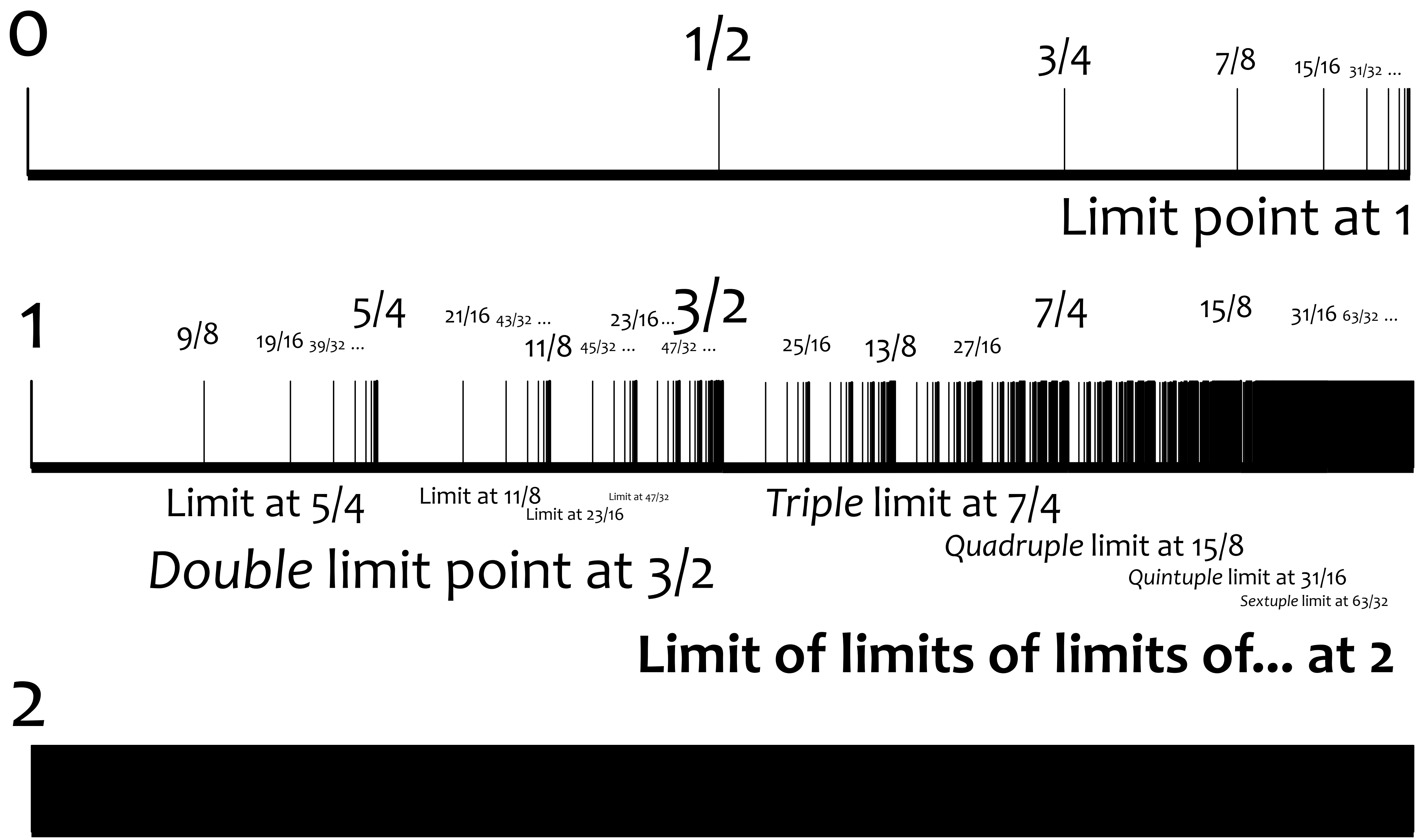
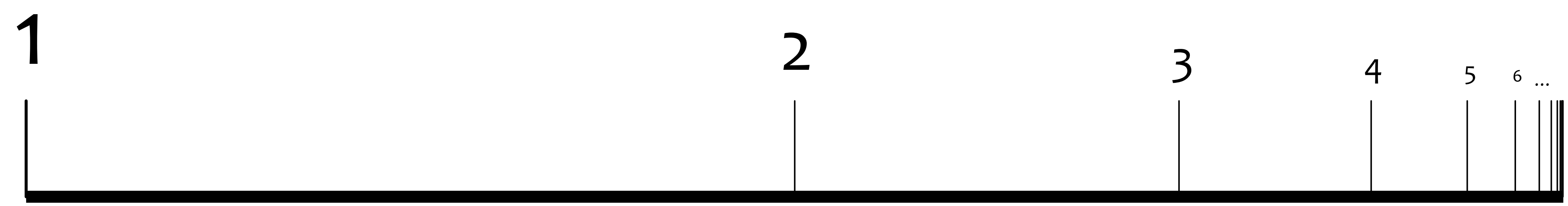


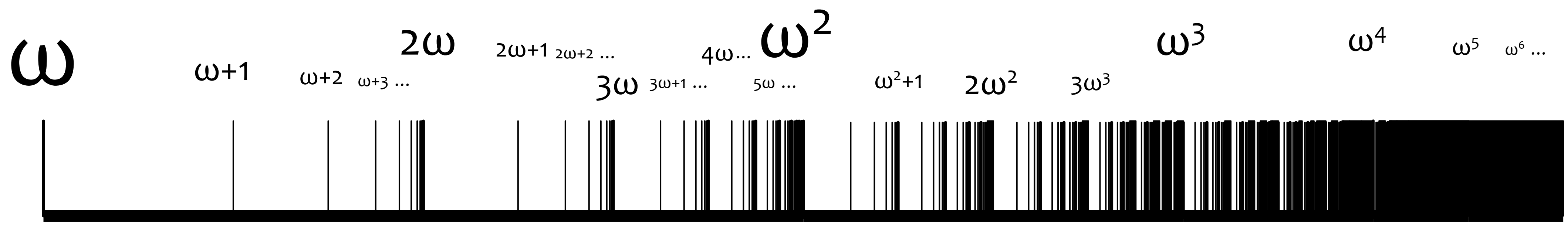


Small examples



Ordinals





- $\mathrm{Ord}(x+1) = \omega^{\mathrm{Ord}(x)}$ for any fusible number x
- $\mathrm{Ord}(n) = \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}n$ for any integer n
- The fusible numbers are *well-ordered*, with order type $\varepsilon_0 = \omega^{\varepsilon_0}$

Margin

- $m(x)$ = difference between x and smallest fusible number > x

$$\text{If } x<0, \text{ then } m(x) = -x$$

$$\text{Otherwise, } m(x) = m(x - m(x-1))/2$$

- Recursive calls give a fuse configuration for smallest fusible > x

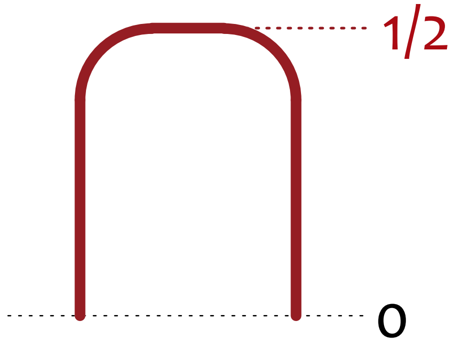
Code!

```
from fractions import *
from sys import *
from string import *
from time import *

def memoize(function):
    cache = {}
    def decorated_function(*args):
        if args in cache:
            return cache[args]
        else:
            val = function(*args)
            cache[args] = val
            return val
    return decorated_function
```

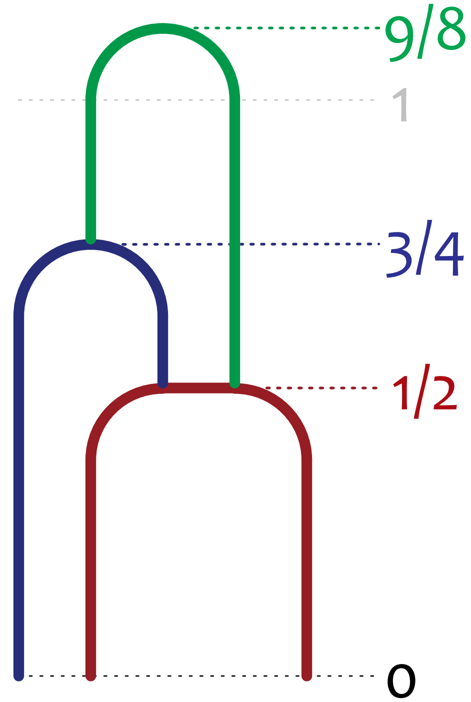
```
@memoize
def margin(x):
    depth = 0
    while (x >= 0):
        x = x - margin2(x-1)
        depth = depth+1
    return -x/(1L<<depth)

def log2den(q):
    return count(bin(q.denominator), '0', 2)

def logmargin(x):
    return log2den(margin(x))
```

Some small margins

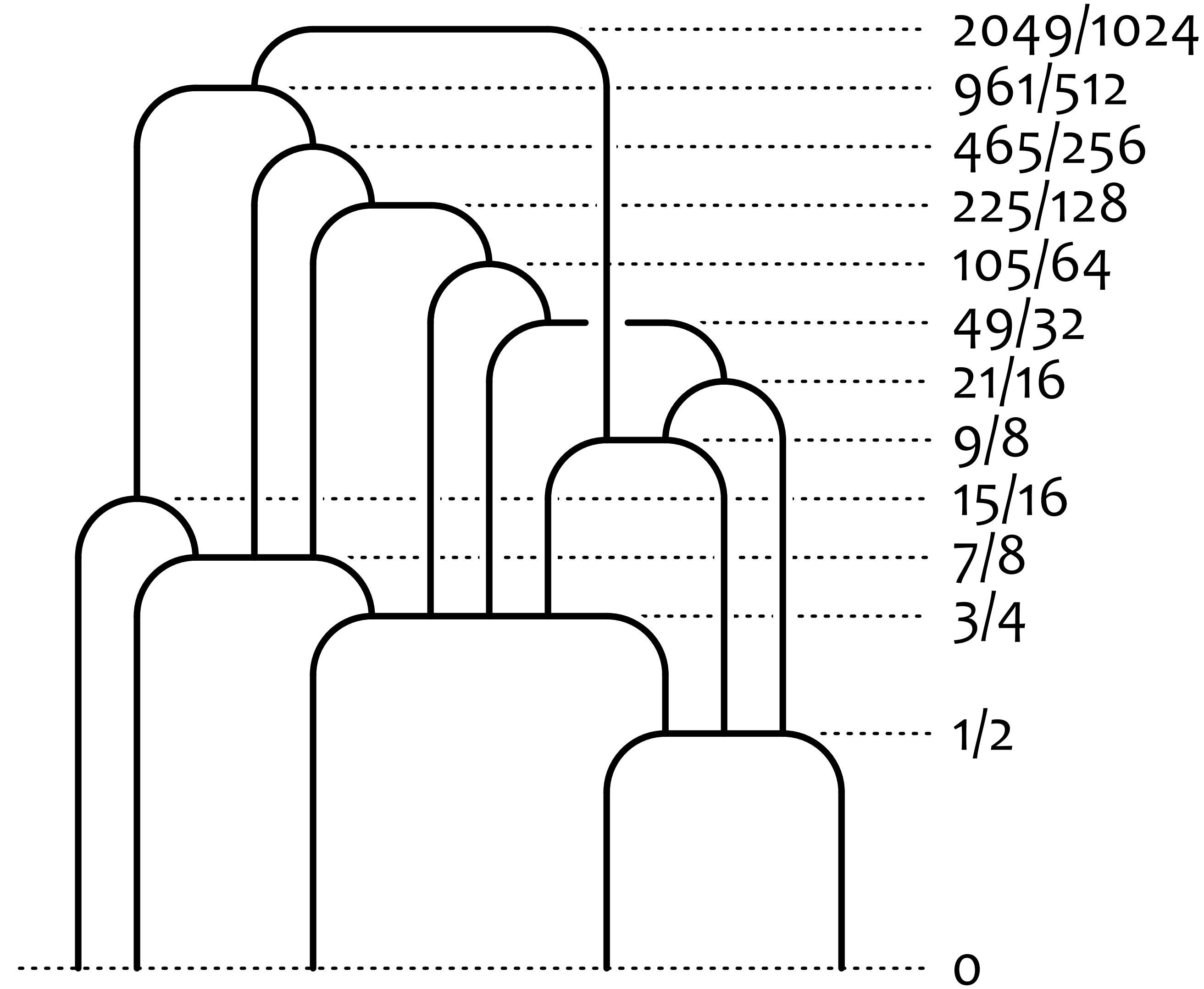
$m(0) = 2^{-1}$



$m(1) = 2^{-3}$



$m(2) = 2^{-10}$



So what's m(3)?

1, 3, 10, … ?

- $-\log_2 m(0) =$ **1**
- $-\log_2 m(1) =$ **3**
- $-\log_2 m(2) =$ **10**
- $-\log_2 m(3) =$ **1,541,023,937**
- $-\log_2 m(4) =$ ***REALLY REALLY BIG!***

  (probably between Skewes' # and Graham's #)

Thanks, Martin!